

A note on the strong consistency of M-estimates in linear models*

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Abstract We improve a known result on the strong consistency of M-estimates of the regression parameters in a linear model for independent and identically distributed random errors under some mild conditions.

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We consider the linear model

$$Y_i = x_i' \beta_0 + e_i, i = 1, 2, \dots, n, \quad (1)$$

where x_i are $p \times 1$ known design vectors, β_0 is a $p \times 1$ unknown vector of regression coefficients and $\{e_i\}$ are error variables. The M-estimate $\hat{\beta}_n$ of β_0 is defined by minimizing

$$\sum_{i=1}^n \rho(Y_i - x_i' \beta), \quad (2)$$

where ρ is a convex function. Important examples include Huber's estimate with $\rho(x) = (x^2 I(|x| \leq c))/2 + (c|x| - c^2/2)I(|x| > c)$, $c > 0$, where $I(A)$ is the indicator function of the set A , the \mathcal{L}^q regression estimate with $\rho(x) = |x|^q$, $1 \leq q \leq 2$, and regression quantiles with $\rho(x) = \rho_\alpha(x) = \alpha x^+ + (1 - \alpha)(-x)^+$, $0 < \alpha < 1$, where $x^+ = \max(x, 0)$. In particular, if $q = 1$ or $\alpha = 1/2$, then the minimizer of (2) is called the least absolute deviation estimate. There is a substantial amount of work concerning asymptotic properties of M-estimates.

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We assume that ρ is a non-monotonic convex function on \mathbb{R} with right and left derivatives ψ_+ and ψ_- . Choose ψ such that $\psi_-(u) \leq \psi(u) \leq \psi_+(u)$ for all $u \in \mathbb{R}$. Write $S_n = \sum_{i=1}^n x_i x'_i$, $d_n = \max_{1 \leq i \leq n} x'_i S_n^{-1} x_i$, and assume that $S_{n_0} > 0$ for some integer n_0 and that $n \geq n_0$.

Zhao (2002) established the following result.

Theorem A. Assume that $\{e_i, i \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables. Suppose there exist positive constants Δ , C_0 , C_1 and $\delta \in (0, 1]$ such that the following conditions are satisfied:

$$\psi(u+h) - \psi(u) \leq C_0 \quad \text{for } h \in (0, \Delta) \quad \text{and } u \in \mathbb{R}, \quad (3)$$

$$E\psi(e_1) = 0, \quad |E\psi(e_1 + u)| \geq C_1|u| \quad \text{for } |u| < \Delta, \quad (4)$$

$$d_n = O(n^{-\delta}).$$

Assume further that $E|\psi(e_1)|^{1/\delta} < \infty$ if $0 < \delta < 1$ and $E|\psi(e_1)|^q < \infty$ for some $q > 1$ if $\delta = 1$. Then $\hat{\beta}_n \rightarrow \beta_0$ a.s as $n \rightarrow \infty$.

We will further discuss the strong consistency of $\hat{\beta}_n$ and obtain the following result.

Theorem 1. In model (1), assume that $\{e_i, i \geq 1\}$ is a sequence of i.i.d. random variables. Suppose that conditions (3) and (4) are satisfied and

$$d_n = O(n^{-1}).$$

Then $\hat{\beta}_n \rightarrow \beta_0$ a.s as $n \rightarrow \infty$.

Proof For the technical proof, see the Appendix.

Remark. In Theorem A, $E|\psi(e_1)|^q < \infty$ for some $q > 1$ is needed for the case $\delta = 1$. Theorem 1 is obtained without the condition $E|\psi(e_1)|^q < \infty$ for some $q > 1$ in the case $\delta = 1$. Theorem 1 improves the result of Theorem A.

Appendix

To prove the main results of the paper, we need the following lemmas.

Lemma 1 (c.f. Bennett, 1962). Assume that $\{X_n, n \geq 1\}$ is a sequence of independent random variables such that $EX_n = 0$ and $|X_n| \leq b$ for all $n \geq 1$ and some $b > 0$. Denote $B_n^2 = \sum_{i=1}^n EX_i^2$. Then for all $\varepsilon > 0$,

$$P\left(\left|\sum_{i=1}^n X_i\right| > \varepsilon\right) \leq 2 \exp\left\{-\frac{\varepsilon^2}{2b\varepsilon + 2B_n^2}\right\}.$$

Lemma 2 (c.f. Choi and Sung, 1987). Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX_1 = 0$. Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of constants. If $\max_{1 \leq i \leq n} |a_{ni}| = O(n^{-1})$, then

$$\sum_{i=1}^n a_{ni} X_i \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof of Theorem 1. Let $x_{ni} = S_n^{-1/2} x_i$, $\beta_{n0} = S_n^{1/2} \beta_0$ and $\hat{\beta}_n^* = S_n^{1/2} \hat{\beta}_n$. Then

$$\sum_{i=1}^n x_{ni} x'_{ni} = I_p, \quad \sum_{i=1}^n \|x_{ni}\|^2 = p, \quad d_n = \max_{1 \leq i \leq n} \|x_{ni}\|^2, \quad (\text{A.1})$$

where I_p is the $p \times p$ identity matrix, and $\|\cdot\|$ is the Euclidean norm on \mathbb{R}^p .

Model (1) can be rewritten as

$$Y_i = x'_{ni} \beta_{n0} + e_i, \quad i = 1, 2, \dots, n \quad (\text{A.2})$$

and

$$\sum_{i=1}^n \rho(Y_i - x'_{ni} \hat{\beta}_n^*) = \min_{\beta} \sum_{i=1}^n \rho(Y_i - x'_{ni} \beta). \quad (\text{A.3})$$

Without loss of generality, we assume that the true parameter $\beta_0 = 0$ in model (1), i.e., $\beta_{n0} = 0$ in (A.2). Denote the unit sphere $U = \{\beta : \beta \in \mathbb{R}^p, \|\beta\| = 1\}$. Let $\varepsilon > 0$ be any given constant. Without loss of generality, it can be assumed that $2C_2\varepsilon < \Delta$. Define

$$D_n(\beta) = \sum_{i=1}^n \{\rho(e_i - x'_{ni} \beta) - \rho(e_i)\}, \quad \beta \in \mathbb{R}^p$$

and

$$D_n(\varepsilon n^{1/2} \gamma) = \sum_{i=1}^n \{\rho(e_i - \varepsilon n^{1/2} x'_{ni} \gamma) - \rho(e_i)\}$$

$$\begin{aligned}
&= \sum_{i=1}^n \int_0^{w'_{ni}\gamma} \{\psi(e_i + t) - \psi(e_i)\} dt + \sum_{i=1}^n w'_{ni}\gamma \psi(e_i), \quad \gamma \in U, \\
&=: I_{1n}(\gamma) + I_{2n}(\gamma),
\end{aligned}$$

where $w_{ni} = -\varepsilon n^{1/2} x_{ni}$. Hence

$$\inf_{\gamma \in U} D_n(\varepsilon n^{1/2} \gamma) \geq \inf_{\gamma \in U} I_{1n}(\gamma) + \inf_{\gamma \in U} I_{2n}(\gamma) \geq \inf_{\gamma \in U} I_{1n}(\gamma) - \sup_{\gamma \in U} |I_{2n}(\gamma)|. \quad (\text{A.4})$$

We can divide U into N parts, U_1, U_2, \dots, U_N , such that the diameter of each part is less than n^{-2} and $N \leq (2n^2 + 1)^p$. Let T_j be the smallest close convex set covering U_j . For a fixed T_j , there are three cases as follows.

i) $w'_{ni}\gamma \geq 0$ for each $\gamma \in T_j$, then there exists a $\gamma_{ij} \in T_j$ such that $w'_{ni}\gamma_{ij} = \inf\{w'_{ni}\gamma : \gamma \in T_j\}$.

ii) $w'_{ni}\gamma \leq 0$ for each $\gamma \in T_j$, then there exists a $\gamma_{ij} \in T_j$ such that $w'_{ni}\gamma_{ij} = \sup\{w'_{ni}\gamma : \gamma \in T_j\}$.

iii) $w'_{ni}\gamma > 0$ for some $\gamma \in T_j$, and $w'_{ni}\gamma < 0$ for some $\gamma \in T_j$, then there exists a $\gamma_{ij} \in T_j$ such that $w'_{ni}\gamma_{ij} = 0$.

Write

$$G(t) = E\psi(e_i + t), \quad \Psi_i(t) = \psi(e_i + t) - \psi(e_i) - G(t), \quad t \in \mathbb{R}.$$

By the monotonicity of ψ ,

$$\begin{aligned}
\inf_{\gamma \in U} I_{1n}(\gamma) &\geq \inf_{1 \leq j \leq N} \inf_{\gamma \in T_j} I_{1n}(\gamma) \geq \inf_{1 \leq j \leq N} \sum_{i=1}^n \int_0^{w'_{ni}\gamma_{ij}} \{\psi(e_i + t) - \psi(e_i)\} dt \\
&\geq \inf_{1 \leq j \leq N} \sum_{i=1}^n \int_0^{w'_{ni}\gamma_{ij}} G(t) dt \left\{ 1 - \frac{|\sum_{i=1}^n \int_0^{w'_{ni}\gamma_{ij}} \Psi_i(t) dt|}{\sum_{i=1}^n \int_0^{w'_{ni}\gamma_{ij}} G(t) dt} \right\}. \quad (\text{A.5})
\end{aligned}$$

Let $\gamma \in U_j$ and $\gamma_{ij} \in T_j$. By (A.1) and the definition of U_j and T_j , for large enough n ,

$$\begin{aligned}
\sum_{i=1}^n (x'_{ni}\gamma_{ij})^2 - \sum_{i=1}^n (x'_{ni}\gamma)^2 &\leq \left| \sum_{i=1}^n (x'_{ni}\gamma_{ij})^2 - \sum_{i=1}^n (x'_{ni}\gamma)^2 \right| \\
&= \left| \sum_{i=1}^n (\gamma_{ij} - \gamma)' x_{ni} x'_{ni} (\gamma_{ij} + \gamma) \right| \\
&\leq \sum_{i=1}^n \|\gamma_{ij} - \gamma\| \|x_{ni}\|^2 (\|\gamma_{ij} - \gamma\| + 2\|\gamma\|) \\
&\leq \sum_{i=1}^n n^{-2} (n^{-2} + 2) \|x_{ni}\|^2 \\
&\leq 3pn^{-2} < 1/2,
\end{aligned}$$

combining (A.1), we obtain that for $1 \leq j \leq N$,

$$\begin{aligned} \sum_{i=1}^n (x'_{ni} \gamma)^2 &\geq \sum_{i=1}^n (x'_{ni} \gamma)^2 - 1/2 = \gamma' \sum_{i=1}^n x_{ni} x'_{ni} \gamma - 1/2 \\ &= \|\gamma\|^2 - 1/2 = 1/2. \end{aligned} \quad (\text{A.6})$$

By (5) and the selection of ε , for large enough n and for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, N$,

$$|w'_{ni} \gamma_{ij}| = |\varepsilon n^{1/2} x'_{ni} \gamma_{ij}| \leq C_2 \varepsilon \|\gamma_{ij}\| \leq C_2 \varepsilon (1 + n^{-2}) < 2C_2 \varepsilon < \Delta.$$

It follows that by (4) and (A.6),

$$\begin{aligned} \inf_{1 \leq j \leq N} \sum_{i=1}^n \int_0^{w'_{ni} \gamma_{ij}} G(t) dt &\geq \inf_{1 \leq j \leq N} C_1 \sum_{i=1}^n \int_0^{w'_{ni} \gamma_{ij}} t dt \\ &= \inf_{1 \leq j \leq N} \frac{C_1}{2} \sum_{i=1}^n (w'_{ni} \gamma_{ij})^2 \\ &\geq \frac{C_1 \varepsilon^2 n}{4}. \end{aligned}$$

For $1 \leq j \leq N$, denote $Y_{ni}^{(j)} = \int_0^{w'_{ni} \gamma_{ij}} \Psi_i(t) dt$. By (3) and (A.6), one has that

$$|Y_{ni}^{(j)}| \leq 2C_0 |w'_{ni} \gamma_{ij}| < 4C_0 C_2 \varepsilon =: C_3.$$

For $1 \leq j \leq N$, we also obtain that by

$$\begin{aligned} \sum_{i=1}^n \text{Var}(Y_{ni}^{(j)}) &\leq \sum_{i=1}^n \mathbb{E} \left\{ \int_0^{w'_{ni} \gamma_{ij}} (\psi(e_i + t) - \psi(e_i)) dt \right\}^2 \\ &\leq C_0 \sum_{i=1}^n |w'_{ni} \gamma_{ij}| \int_0^{w'_{ni} \gamma_{ij}} G(t) dt \\ &\leq \frac{C_3}{2} \sum_{i=1}^n \int_0^{w'_{ni} \gamma_{ij}} G(t) dt. \end{aligned}$$

Define event A_n by

$$A_n := \left\{ \sup_{1 \leq j \leq N} \frac{|\sum_{i=1}^n \int_0^{w'_{ni} \gamma_{ij}} \Psi_i(t) dt|}{\sum_{i=1}^n \int_0^{w'_{ni} \gamma_{ij}} G(t) dt} \geq \frac{1}{2} \right\}.$$

Noting the fact that $N \leq (2n^2 + 1)^p$ and using Lemma 1, we have that

$$P(A_n) \leq \sum_{j=1}^N P\left(\left|\sum_{i=1}^n Y_{ni}^{(j)}\right| \geq \frac{1}{2} \sum_{i=1}^n \int_0^{w'_{ni} \gamma_{ij}} G(t) dt\right)$$

$$\begin{aligned}
&\leq 2 \sum_{j=1}^N \exp \left\{ - \frac{\frac{1}{4} (\sum_{i=1}^n \int_0^{w'_{ni} \gamma_{ij}} G(t) dt)^2}{2C_3 \times \frac{1}{2} \sum_{i=1}^n \int_0^{w'_{ni} \gamma_{ij}} G(t) dt + 2C_3 \times \frac{1}{2} \sum_{i=1}^n \int_0^{w'_{ni} \gamma_{ij}} G(t) dt} \right\} \\
&= 2 \sum_{j=1}^N 2 \exp \left\{ - \frac{1}{8C_3} \sum_{i=1}^n \int_0^{w'_{ni} \gamma_{ij}} G(t) dt \right\} \\
&\leq 2(2n^2 + 1)^p \exp(-C_4 n),
\end{aligned}$$

thus $\sum_{n=1}^{\infty} P(A_n) < \infty$. By Borel-Cantelli lemma, it follows that $P(A_n, i.o.) = 0$. Hence with probability one for large enough n ,

$$\sup_{1 \leq j \leq N} \frac{\sum_{i=1}^n \int_0^{w'_{ni} \gamma_{ij}} \Psi_i(t) dt}{\sum_{i=1}^n \int_0^{w'_{ni} \gamma_{ij}} G(t) dt} < \frac{1}{2},$$

which implies by (A.5) that with probability one for large enough n ,

$$\inf_{\gamma \in U} I_{1n}(\gamma) \geq \frac{C_1 \varepsilon^2 n}{8}. \quad (\text{A.7})$$

Denote

$$x_{ni} = \begin{pmatrix} x_{ni1} \\ x_{ni2} \\ \vdots \\ x_{nip} \end{pmatrix}, \gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_p \end{pmatrix}, \quad a_{ni} = \begin{cases} \frac{x_{nik}}{n^{1/2}}, 1 \leq i \leq n \\ 0, i \geq n \end{cases} \quad \text{for fixed } k = 1, 2, \dots, p.$$

By (A.1) and $d_n = O(n^{-1})$, for fixed $k = 1, 2, \dots, N$, $|x_{nik}| \leq \|x_{ni}\| \leq d_n^{1/2} \leq \sqrt{C_2} n^{-1/2}$ and $\sum_{i=1}^n x_{nik}^2 = 1$, thus

$$|a_{ni}| \leq \sqrt{C_2} n^{-1} \quad \text{for } n \geq 1.$$

Using Lemma 2, it follows that

$$n^{-1/2} \sum_{i=1}^n x_{nik} \psi(e_i) = \sum_{i=1}^n a_{ni} \psi(e_i) \rightarrow 0 \quad a.s.$$

Hence,

$$\begin{aligned}
n^{-1} \sup_{\gamma \in U} |I_{2n}(\gamma)| &= n^{-1} \sup_{\gamma \in U} \left| \sum_{i=1}^n w'_{ni} \gamma \psi(e_i) \right| = n^{-1} \sup_{\gamma \in U} \left| \sum_{i=1}^n \varepsilon n^{1/2} x'_{ni} \gamma \psi(e_i) \right| \\
&= \varepsilon n^{-1/2} \sup_{\gamma \in U} \left| \sum_{i=1}^n \sum_{k=1}^p x_{nik} \gamma_k \psi(e_i) \right| \\
&= \varepsilon n^{-1/2} \sup_{\gamma \in U} \left| \sum_{k=1}^p \left(\sum_{i=1}^n x_{nik} \psi(e_i) \right) \gamma_k \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon n^{-1/2} \sup_{\gamma \in U} \sqrt{\sum_{k=1}^p \left(\sum_{i=1}^n x_{nik} \psi(e_i) \right)^2} \sqrt{\sum_{k=1}^p \gamma_k^2} \\
&= \varepsilon \sqrt{\sum_{k=1}^p \left(n^{-1/2} \sum_{i=1}^n x_{nik} \psi(e_i) \right)^2} \rightarrow 0 \quad a.s.,
\end{aligned}$$

which implies that with probability one for large enough n ,

$$\sup_{\gamma \in U} |I_{2n}(\gamma)| \leq \frac{C_2 \varepsilon^2 n}{16}. \quad (\text{A.8})$$

Combining (A.4) with (A.7) and (A.8), we have that with probability one for large enough n ,

$$\inf_{\gamma \in U} D_n(\varepsilon n^{1/2} \gamma) \geq \frac{C_2 \varepsilon^2 n}{16}.$$

By the convexity of $D_n(\cdot)$, $D_n(0) = 0$ and the definition (A.3) of $\hat{\beta}_n^*$, it follow that

$$\left\{ \inf_{\gamma \in U} D_n(\varepsilon n^{1/2} \gamma) > 0 \right\} \subset \{ \|\hat{\beta}_n^*\| \leq \varepsilon n^{1/2} \}.$$

Thus for any given $\varepsilon > 0$, with probability one for large enough n ,

$$\|\hat{\beta}_n^*\| \leq \varepsilon n^{1/2},$$

which implies that

$$n^{-1/2} \|\hat{\beta}_n^*\| \rightarrow 0 \quad a.s. \quad \text{as } n \rightarrow \infty.$$

Denote $\zeta(A)$ for the smallest eigenvalue of a positive definite matrices A . Based on the result that if A and B are two positive definite matrices of order p , then

$$\text{tr}(AB) \geq \mu(A)\zeta(B),$$

where $\mu(A)$ is the largest eigenvalue of A (see Chen & Zhao, 1995). Now we take $M = n_0$ and $n > n_0$ so that $S_n \geq S_M > 0$. By (5), one has that

$$\begin{aligned}
\zeta(S_M)(\zeta(S_M))^{-1} &\leq \zeta(S_M) \text{tr}(S_n^{-1}) \leq \text{tr}(S_M^{1/2} S_n^{-1} S_M^{1/2}) \\
&= \text{tr}(S_n^{-1} S_M) = \sum_{i=1}^M x_i' S_n^{-1} x_i \leq M d_n \leq M C_2 n^{-1},
\end{aligned}$$

and there exists a positive constant C_5 such that $1 \leq C_5 n^{-1/2} \zeta(S_n^{1/2})$. It follows that

$$\|\hat{\beta}_n\| \leq C_5 n^{-1/2} \|S_n^{1/2} \hat{\beta}_n\| = C_5 n^{-1/2} \|\hat{\beta}_n^*\| \rightarrow 0 \quad a.s. \quad \text{as } n \rightarrow \infty.$$

The proof of the theorem is completed.

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